Quadratic Involutions on the Plane Rational Quartic

Dissertation

Submitted to the Board of University Studies of the Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

By

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1911

Section I

The General Theory of Involution Curves of a Plane Rational Curve of Order n.

Let R^n denote a plane rational curve of order n, and let it be given by the equation

$$(1)\quad \begin{cases} x_0 = a_0 t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots\cdots + a_n \\ x_1 = b_0 t^n + b_1 t^{n-1} + b_2 t^{n-2} + \cdots\cdots + b_n \\ x_2 = c_0 t^n + c_1 t^{n-1} + c_2 t^{n-2} + \cdots\cdots + c_n . \end{cases}$$

If we join the parameters t_1 and t_2 by a line, where t_1 and t_2 are in an involution of the form

$$(2)\quad A t_1 t_2 + B(t_1 + t_2) + C = 0,$$

we shall show that the locus of this line is a rational curve of class $n-1$ which touches R^n $2(n-2)$ times and meets it in $2(n-2)(n-3)$ other points. This class curve will be called an involution curve, and will be denoted by R^{n-1}.

Cut the curve R^n by any line

(3) $(\xi y) \equiv \xi_0 y_0 + \xi_1 y_1 + \xi_2 y_2 = 0$

and we have

(4) $(\alpha_0 \xi)t^n + (\alpha_1 \xi)t^{n-1} + \dots + (\alpha_n \xi) = 0.$

For convenience suppose we choose the involution with 0 and ∞ as double points. Then t and $-t$ must satisfy the last equation and we have for n even, say $n=2m$,

(5) $(\alpha_0 \xi)t^{2m} + (\alpha_1 \xi)t^{2m-1} + \dots + (\alpha_n \xi) = 0$

(6) $(\alpha_0 \xi)t^{2m} - (\alpha_1 \xi)t^{2m-1} + \dots + (\alpha_n \xi) = 0$

Whence by addition and then subtraction we get

(7) $(\alpha_0 \xi)t^{2m} + (\alpha_2 \xi)t^{2m-2} + \dots + (\alpha_n \xi) = 0$ and

(8) $(\alpha_1 \xi)t^{2m-2} + (\alpha_3 \xi)t^{2m-4} + \dots + (\alpha_{n-1} \xi) = 0$

Since only even powers occur we can divide the exponent by 2 and write

(9) $(\alpha_0 \xi)t^{m} + (\alpha_2 \xi)t^{m-1} + \dots + (\alpha_n \xi) = 0$

(10) $(\alpha_1 \xi)t^{m-1} + (\alpha_3 \xi)t^{m-2} + \dots + (\alpha_{n-1} \xi) = 0$

Eliminating ξ_0, ξ_1, ξ_2 from equations (3), (8), and (9) we have the locus required in determinant form,

$$(1) \quad \begin{vmatrix} x_0 , & x_1 , & x_2 \\ f_0(t^m) , & f_1(t^m) , & f_2(t^m) \\ f_0'(t^{m-1}) , & f_1'(t^{m-1}) , & f_2'(t^{m-1}) \end{vmatrix} = 0$$

or written parametrically its equation is

$$\xi_i = F_i(t^{2m-1}) ,$$

and since $n = 2m$ we have as the representation of the involution curve

$$(2) \quad \xi_i = F_i(t^{n-1}) ,$$

which is a rational class curve of order $n-1$. Similar argument holds for n odd.

The point to be emphasized is that the parameter may be replaced by a new one which reduces the degree by one half, that is $t^2 = \tau$, defines a quadratic [illegible]

This accounts then for there being a triple infinity of ways depending on the values of $\alpha, \beta, \lambda, \delta$ in a transformation of the form

$$t = \frac{\alpha t + \beta}{\lambda t + \delta}$$

If the double points of the involution are given by the quadratic $(at)^2$ then we choose any two quadratics apolar to $(at)^2$, say $(\alpha t)^2$ and $(\beta t)^2$, and any convenient member of the pencil $(\alpha t)^2 + \lambda(\beta t)^2$ will serve as the new parameter t. In the case above we had the double points given by

$$t = 0$$

And two quadratics apolar to it are

$$\alpha t^2 + \beta = 0$$

and $\lambda t^2 + \delta = 0$

The new parameter are

$$T = \alpha t^2 + \beta + \lambda(\gamma t^2 + \delta).$$

In particular, if we choose $\alpha = 1$, $\beta = -2$, $\gamma = 1$, $\delta = 1$, $\lambda = -2$ we have

$$T = t^2$$

To find the number of contacts of the R^{n-1} with the R^n we consider first the case of the R^4 and its involution cubic R^3. The R^4 is of class six so there are eighteen common lines. There are three ways in which we may have common lines. Suppose a line meets the R^4 in four points whose parameters are [illegible] t_1, t_2, t_3, t_4. The three cases in which common lines occur are:

1) When t_1 and t_2 are a pair of the involution.

2) When t_3 and t_4 are a pair of the involution, and

3) When t_1 and t_3 are a pair of the involution.

Case 1) can happen only twice, that is when the line cuts out the double points of the involution. This accounts for two common lines.

In Case 2) a tangent at T meets the curve again (at t say). For a given T there are two t's, and since the curve is of class six for a given t there are four T's, corresponding to the four tangents from the point on the curve. The relation connecting T and t is

$$f(T, t^2) \equiv f_1(T^4)t^2 + f_2(T^4)t + f_3(T^4) = 0$$

The condition that the roots t

be an involution is of the fourth degree in t, which means there are four common lines for this case.

(Now case 3) must contain all the other common lines which is twelve. This case arises when t_1 and t_3 are a pair of the involution, but t_2 and t_3 are as well a pair of the involution; therefore the twelve common lines are six repeated. In other words the R^3 has six contacts with the R^4.

This is easily extended to the general case of R^n. The R^n is of class $2(n-1)$, so the R^n and the R^{n-1} have $2(n-1)(n-1)$ common lines.

There will always be two of these

accounted for in case 1), correspond-ing to the two double points of the involution.

For case 2) the equation con-necting a point of tangency T and a point of intersection of the tangent at t is of degree $2n-4$ in T and $n-2$ in t, i.e.

$$f(T^{2n-4}, t^{n-2}) = 0$$

For two values of t to be in an involution is a condition of degree $n-3$ in the coeffi-cients of t, and hence of degree $2(n-2)(n-3)$ in T. Therefore there are $2(n-2)(n-3)$ common lines for case 2).

Subtracting the common lines for case 1) and case 2) from the total number we have for case 3)

$2(n-1)(n-1) - 2(n-2)(n-3) - 2 = 6n - 12$

But since these pairs [illegible] have general $3n-6$ contacts of R^n and r^{n-1}.

The r^{n-1} is of order $2n-4$, hence the R^n and its involution curve intersect in $2n(n-2)$ points. The contacts count for $6n-12$ intersections, so there are $2(n-2)(n-3)$ remaining intersections.

If the parameters of a node of R^n are in the involution, then the node is a factor of the involution curve and the remaining factor is an r^{n-2}. The first factor of the locus, being a double point of the R^n, will count for two contacts, hence the remaining factor, an r^{n-2}, will

have only $3n-8$ contacts).

If $n-3$ nodes are made a part of the locus then the conditions set for the remaining part of the involution curve are $\infty^{(n-1-n+3-2)}$ that is an ∞^2 with n contacts, or a conic all of whose intersections with the R^n are contacts. It must be remembered however that two sets of the involution determine the involution, therefore for n greater than five, it is $n-5$ conditions on the R^n for $n-3$ nodes to be in the involution.*

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* A case in point would be the unique rational sextic with its nodes at the ten points of a Desargues configuration B. Any three nodes on a line would be in an involution, hence we could get ten conics having full contact with the sextic

We shall now consider the R^3 and find the involution conic. We found that there are in general $3n-6$ contacts, so in this case we get three contacts and no extra intersections.

Let one flex of the R^3 be 0 and another ∞, and take these two flex tangents and the line joining the flexes as reference triangle. Then the equation of R^3 is

$$x_0 = a_0t^3 + b_0t^2$$

$$1) \quad x_1 = c_1t + d_1$$

$$x_2 = e_2t^2 - f_2t$$

If 0 and ∞ are the double points of the involution it is of the form

$$2) \quad t_1 + t_2 = 0,$$

and the line whose locus is the involution conic will join t and $-t$

of the R^3. Such a line is given by

$$(3)\qquad \begin{vmatrix} \xi_0 & \xi_1 & \xi_2 \\ a_0 t^3 + b_0 t^2 & c_1 t + d_1 & b_2 t^2 + c_2 t \\ -a_0 t^3 + b_0 t^2 & -c_1 t + d_1 & b_2 t^2 - c_2 t \end{vmatrix} = 0$$

This determinant is readily seen to reduce to the following:

$$(4)\qquad \begin{vmatrix} \xi_0 & \xi_1 & \xi_2 \\ b_0 t^2 & d_1 & b_2 t^2 \\ a_0 t^2 & c_1 & c_2 \end{vmatrix} = 0$$

which is parametrically

$$(5)\qquad \begin{aligned} \xi_0 &= -b_2 c_1 t^2 + c_2 d_1, \\ \xi_1 &= a_0 b_2 t^4 - b_0 c_2 t^2 \\ \xi_2 &= (b_0 c_1 - a_0 d_1) t^2 \end{aligned}$$

It is seen that only even powers of t occur, so we replace the parameter t^2 by a new one, t' say. For convenience we drop the prime and write the equation in the form

$$\xi_0 = -b_2 c_1 t - c_2 d_1$$
(6) $$\xi_1 = a_0 b_2 t^2 - b_0 c_2 t$$
$$\xi_2 = (b_0 c_1 - a_0 d_1) t$$

This is the involution conic in line form and we want it in point form. We have

(7) $$\begin{vmatrix} \xi_0 & , \; \xi_1 & , \; \xi_2 \\ -b_2 c_1 t + c_2 d_1 & a_0 b_2 t^2 - b_0 c_2 t & , \; (b_0 c_1 - a_0 d_1) t \\ -b_2 c_1 & , \; 2 a_0 b_2 - b_0 c_2 & , \; b_0 c_1 - a_0 d_1 \end{vmatrix} = 0$$

which is in x_i's

$$x_0 = a_0 b_2 (a_0 d_1 - b_0 c_1) t^2$$
(8) $$x_1 = c_2 d_1 (a_0 d_1 - b_0 c_1)$$
$$x_2 = -a_0 b_2^2 c_1 t^2 + 2 a_0 b_2 c_2 d_1 t - b_0 c_2^2 d_1$$

Now in order to get the intersections of this involution conic with the R^3 we must eliminate the parameter from the equation of the conic and thus get an equation of the second degree in t

If we then substitute for the x_i's their values in the equation of the R^3 we obtain a sextic in t which will give the intersections of the two curves.

Eliminating t from (8) we get

$$(9)\quad b_2^2 c_1^2 x_1^2 + b_0^2 c_2^2 x_1^2 - (a_0^2 d_1^2 - 2a_0 b_0 c_1 d_1 - b_0^2 c_1^2) x_2^2$$
$$+ 2(a_0 b_0 c_2 d_1 - b_0^2 c_1 c_2) x_1 x_2 + 2(a_0 b_2 c_1 d_1 - b_0 b_2 c_1^2) x_0 x_2$$
$$+ 2(b_0 b_2 c_1 c_2 - a_0 b_2 c_2 d_1) x_0 x_1 = 0$$

If we now substitute for the x's their values in equation (1) we get

$$(10)\quad a_0^2 b_2^2 c^2 t^6 + 2a_0^2 b_2^2 c\, d\, t^5 + (a_0^2 b_2^2 d^2 - 2a_0 b_0 b_2 c_1 c_2 d) t^4$$
$$- (2a_0^2 b_2 c_2 d^2 - 2a_0 b_0 b_2 c_1 c_2 d) t^3$$
$$+ (a_0^2 c_2^2 d^2 - 2a_0 b_0 b_2 c_2 d^2) t^2 - 2a_0 b_0 c_2^2 d^2 t$$
$$+ b_0^2 c_2^2 d^2 = 0$$

This sextic is seen to be the square of the cubic

$$(11)\quad a_0 b_2 c\, t^3 + a_0 b_2 d\, t^2 - a_0 c_2 d\, t - b_0 c_2 d = 0$$

which gives the parameters of the three points of contact of the R^3 and its involution conic.

We shall now prove that <u>the points of contact of an R^3 and its involution conic are the points of the Hessian of the cubic giving the parameters of the planes of R^3, and the Hessian points are the double points of the involution.</u>
We shall consider the R^3 given by equation (1), and the involution whose double points are 0 and ∞.
The cubic giving the planes is the fundamental cubic, that is a unique cubic apolar to each of the three binary cubics in (1).
Calculating that cubic in the usual

Then we have

(2) $a_0 b_2 c_1 t^3 + 2 a_0 b_2 d_1 t^2 - 2 a_0 c_2 d_1 t + b_0 c_2 d_1 = 0$

the quadratic giving the double points of the involution under consideration (2)

(3) $t = 0$

The Jacobian of (2) and (3) is

$$\begin{vmatrix} a_0 b_2 c_1 t^2 + 2 a_0 b_2 d_1 t + a_0 c_2 d_1 , & a_0 b_2 d_1 t^2 - 2 a_0 c_2 d_1 t + b_0 c_2 d_1 \\ 1 & t \end{vmatrix}$$

which when developed gives

(4) $a_0 b_2 c_1 t^3 + a_0 b_2 d_1 t^2 - a_0 c_2 d_1 t - b_0 c_2 d_1 = 0$

And is just the same cubic as (1) which gives the points of contact of the R^3 and its involution curves, which proves the Theorem.

This theorem is easily proved symbolically. Let the cubic be given by

(1) $$(\alpha t)^3 = 0$$

and the double points of the involution by

(2) $$(\beta t)^2 = 0$$

Let a pair of the involution be t_1 and t_2 the roots of

(3) $$(m t)^2 = 0$$

The line which cuts out t_1 and t_2, which of course is a line of the involution conic, will meet the R^3 again at a point T. This line will be given by

(4) $$(m t)^2 (t - T) = 0$$

Every line section of R^3 is apolar to the fundamental cubic (1). We have thus

(5) $$(\alpha m)^2 (\alpha T) = 0$$

Also (3) must be apolar to (2) hence

(6) $(a m)^2 = 0$

Again when T is t or t_0 we have 2-contact so we have

(7) $(m T)^2 = 0$

If we put in actual coefficients in (5), (6), and (7) and eliminate m_0, m_1, and m_2 we get

$$\begin{vmatrix} \alpha_0 T + \alpha_1 & \alpha_1 T + \alpha_2 & \alpha_2 T + \alpha_3 \\ a_0 & a & a_2 \\ 1 & -T & T^2 \end{vmatrix} = 0$$

which when developed is

(8) $(\alpha_0 a_1 - \alpha_1 a_0)T^3 + (\alpha_0 a_2 - 2\alpha_2 a_0 + \alpha_1 a_1)T^2$
$-(\alpha_2 a_1 - 2\alpha_1 a_2 + \alpha_3 a_0)T + \alpha_2 a_2 - \alpha_3 a_1 = 0$

This cubic gives the points of contact of the R^3 and the R^2.

The Jacobian of the flexcubic and the double points of the involution is

(9) $(\alpha a)(\alpha t)^2(a t) = 0$

which with actual coefficients is just (8).

We shall next consider R^4 and its involution cubic s^3. The number of contacts are found to be in general $3n-6$ which is just the number of flexes of an R^n. Since the Jacobian of the flex cubic and the quadratic of the double points of the involution gave the contacts of R^3 and s^2, it seems natural to look for some such relation in the case of R^n. The Jacobian of the flex equation in general and the quadratic giving the roots of the involution will always be of the right degree in t, $3n-6$, to give the points of contact of R^n and s^{n-1}. But in the case of R^4 we find the degree in the coefficients

not the same as those of the contact equation. We shall find by a symbolic method the degree of the contact equation in the coefficients of the fundamental involution as well as in the coefficients of the quadratic of the involution.

Suppose the fundamental involution of R^4 is given by

$$(\alpha t)^4 + \lambda(\beta t)^4 = 0$$

Let the double points of the involution be $(\Omega t)^2 = 0$, and let one set of the involution t_1 and t_2 be given by a_t^2.

Let the line on t_1 and t_2 meet the R^4 again at T_1 and T_2.

Since every line section is apolar to the fundamental involution we have

(2) $|\alpha a|^2(\alpha T_1)(\alpha T_2) = 0$. and
(3) $|\beta a|^2(\beta T_1)(\beta T_2) = 0$

Eliminating T_2 from (2) and (3) we get

(4) $|\alpha\beta||\alpha a|^2|\beta a|^2(\alpha T_1)(\beta T_1) = 0$

Since every pair of the involution is apolar to $(at)^2$ we have

(5) $|a Q|^2 = 0$.

Again since when t_1 or t_2 is T_1 we have a coincidence we have

(6) $(a T_1)^2 = 0$.

Solving for the a's in (5) and (6) we find them to be of the first degree in the Q's and of the second degree in the T's. If these values of the a's are put in (4) we get an equation of the first degree in the determinants of the fundamental involution of the

second degree in the Q's and of the sixth degree in T. This is the contract equation.

The flex sextic is the first transvectant of the fundamental involution and is of the first degree in the determinants of the fundamental involution. The Jacobian of the flex sextic F and the quartic Q giving the roots of the involution is a sextic T_1 which is of the first degree in the determinants of the fundamental involution but only of the first degree in the Q's. Taking the Jacobian of T_1 and Q we get a sextic T_2 which is of second degree in the Q's and the first degree in the determinants of the fundamental involution.

Now we propose to show that R, the sextic giving the points of contact of R^4 and S^3 can be built from F, Q, T_2, Δ, and φ where F, Q, T_2 have the meaning just given and where Δ is the discriminant of Q, and φ is the third transvectant of the two members of the fundamental involution. The possible combinations that are of the same degree as R are easily seen. We shall show that

$$R = \lambda \Delta F + \mu T_2 + \nu \varphi Q^2.$$

Let the R^4 be referred to two flex tangents and the line joining these flexes whose parameters are 0 and ∞. Its parametric equation will then be

$$(1)\quad \begin{cases} z_0 = at^4 + bt^3 + ct^2 \\ z_1 = bt^3 + ct^2 + dt \\ z_2 = ct^2 + dt + e \end{cases}$$

We shall choose the involution whose sets are t and $-t$, hence whose double points are given by

(2) $Q \equiv 2t = 0$.

Since we are not interested in the equation of R^3, we proceed to find the equation giving the points of contact with the R^2.

Calculating the fundamental involution of R^2, that is two quartics which are apolar to the three binary quartics in (1) we get

(3) $bct^4 - 6bet^2 - 4cet = 0$ and

(4) $4aet^3 + 6adt^2 - cd = 0$.

The polarized form of (3) and (4) that is where B refers to t_1, t_2, t_3, t_4, is

(5) $bcS_3 - beS_2 - ceS_1 = 0$ and

(6) $acS_3 + adS_2 - cd = 0$

which is known to be the condition that four points be on a line.

Now let two of the t's, say t_1 and t_2 be equal to t, and let σ refer to t_3 and t_4. Then (5) and (6) become

(7) $(bct^2 - be)\sigma_2 - (2bet + ce)\sigma_1 - bet^2 - 2cet = 0$

(8) $(2act + ad)\sigma_2 + (act^2 + 2adt)\sigma_1 + adt^2 - cd = 0$

Taking also the equation

(9) $$\sigma_2 - T\sigma_1 + T^2 = 0$$

and eliminating the σ's from (7), (8) and (9) we have

(10) $$\begin{vmatrix} bct^2 - be, & -2bet - ce, & -bet^2 - 2cet \\ 2act + ad, & act^2 + 2adt, & adt^2 - cd \\ 1, & -T & T^2 \end{vmatrix} = 0$$

This is an equation which is obviously the equation giving the parameters of the six flexes

when T is t, and giving the parameters of the six points of contact of R^4 and Q^3 when T is $-t$.

Putting $T = t$ and developing we get

(1) $F \equiv abc^2t^6 + 3abcdt^5 + 6abcet^4 + (3ace^2 - bc^2d)t^3 - 6acdet^2 + 3bcdet + c^2de = 0$

If $T = -t$, (10) becomes

(2) $\bar{Q} \equiv abc^2t^6 + abcdt^5 + 2abcet^4 + bcdt^3 + 2acdet^2 + bcdet - c^2de = 0$

The Jacobian of F and Q is

(3) $J_1 \equiv abc^2t^6 + 2abcdt^5 + 2abcet^4 - 2acdet^2 - 2bcdet - c^2de = 0$

The Jacobian of J_1 and Q is

(4) $J_2 \equiv 3abc^2t^6 + 2abcdt^5 + 2abcet^4 + 2acdet^2 + 4bcdet + 3c^2de = 0$

The quadratic f is the third transvectant of the members of the fundamental involution. We have found the fundamental involution

for the x^2 under consideration to be (3) and (4). Taking the third transvectant of these two forms we have

(5) $g \equiv 3abc\,x\xi^2 + (2a^2c + 2c^2d)\xi + 3acde = 0$.

Forming the product of g and G^2 we get

(6) $gG^2 \equiv (2abcx\xi^2 + (3ac^2x + 4c^2d)\xi + 2acde^2 = 0$

The discriminant of G is

(7) $\Delta = 1$

Writing down now

$K = \lambda \Delta F + \mu J_2 + \nu g G^2$

we find that K is given for

$\lambda = -\frac{1}{5}, \ \mu = \frac{2}{5}, \ \nu = \frac{1}{5}$.

So to avoid fractions we have finally

(8) $5K = gG^2 + 2J_2 - \Delta F$.

If two nodes of R^3 are in the involution then two nodes will be factor of the R^3 and the remaining factor is some other point. We shall show that the remaining factor of R^3 is the third node.

Let the parameters of one node be given by t^2+a, a second by t^2+b, and the third by a general quadratic $c_0t^2+c_1t+c_2$. The R^3 referred to its nodes has the equation

$$(1)\quad \begin{cases} x_0 = (t^2+a)(c_0t^2+c_1t+c_2) \\ x_1 = (t^2+b)(c_0t^2+c_1t+c_2) \\ x_2 = (t^2+a)(t^2+b) \end{cases}$$

The sets of the involution are, if two nodes are in it, t and $-t$.

The equation of a line joining t and $-t$ is given by the determinant

$$(2)\quad \begin{vmatrix} z_0 & z_1 & z_2 \\ (t^2+a)(c_0t^2+c_1t+c_2), & (t^2+b)(c_0t^2+c_1t+c_2), & (t^2+a)(t^2+b) \\ (t^2+a)(c_0t^2-c_1t+c_2), & (t^2+b)(c_0t^2-c_1t+c_2), & (t^2+a)(t^2+b) \end{vmatrix} = 0$$

If we now take the sum of the second and third rows for a new second row, and their difference for a new third row and remove the factor c_1t from the third row we have

$$(3)\quad \begin{vmatrix} z_0 & , \quad z_1 & , \quad z_2 \\ (t^2+a)(c_0t^2+c_2), & (t^2+b)(c_0t^2+c_2), & (t^2+a)(t^2+b) \\ t^2+a & , \quad t^2+b & \quad 0 \end{vmatrix} = 0$$

Replacing t^2 by T and expressing this equation in terms of ξ_i's we have, after removing common factors,

$$(4)\quad \begin{cases} \xi_0 = -(T+b) \\ \xi_1 = T+a \\ \xi_2 = 0 \end{cases}$$

which shows the other node to be the root of ξ_1^3.

Section II

Involutions Determined by Two Double Lines of the Plane Rational Quartic.

If lines are drawn on the meet of any two double lines of the rational quartic we obtain a quadratic involution I_2. That is to say, where such a line meets the curve in four points the parameters pair off, t_1 and t_2 say.

By choosing $t = 0$ and ∞ as the points of contact of one double tangent we may write the curve

$$(1)\quad \begin{cases} x_0 = 4t^2 \\ x_1 = (a_1t^2+2b_1t+c_1)^2 \\ x_2 = (a_2t^2+2b_2t+c_2)^2 \end{cases}$$

Any line on the meet of x_0 and x_1 will be of the form

$$(2)\quad x_0 - \lambda^2 x_1 = 0, \text{ or}$$

$$(3)\quad 4t^2 - \lambda^2(a_1t^2+2b_1t+c_1)^2 = 0$$

which breaks into factors

$$(4)\quad [2t - \lambda(a_1t^2+2b_1t+c_1)][2t+\lambda(a_1t^2+2b_1t+c_1)] = 0$$

If t_1 is a root of either factor we obtain a value of λ which when substituted back in that factor gives an I_2, and the I_2 is the same for both factors.

Let t be a root of the first factor, then

$$(5) \qquad \lambda = \frac{2t_1}{a t_1^2 + 2h t_1 + c}.$$

Hence we have after removing the factor $t - t_1$,

$$(6) \qquad I_{(0,1)} \equiv a t t_2 - c = 0$$

and is the quadratic involution from the meet of z_0 and z_1.

We shall denote the double lines by 0, 1, 2, 3 and $I_{(i,j)}$ will denote the quadratic involution obtained by drawing lines on the meet of the double lines i and j.

We shall show that the double

points

$J_1 J_0$ are given by the points of contact of the two remaining tangents from the meet of the double lines 0 and 1. The double points of $J_{0,1}$ are given by

[illegible] $J_1 = 0$.

It is easily seen that the Jacobian of the quadratics which give the points of contact of the double lines must give the points of contact of some tangents from the meet of the two double lines. It cannot be either of the double tangents*, so it must be the points of contact of the two remaining tangents from their meet. Forming the Jacobian of the double lines 0 and 1 we have

* as the Jacobian of two squared quadratics is the product of the quadratics and their Jacobian and we have account for all the tangents from the meet of two double lines

$$\begin{vmatrix} a_1 t + b_1 , & b_1 t + c_1 \\ 1 , & t \end{vmatrix} = 0 ,$$

which when developed is

(9) $\quad a_1 t^2 - c_1 = 0$

and gives precisely the roots of the involution $I_{0,1}$.

We shall now prove that <u>the points of contact of the two tangents that may be drawn to the quartic from the meet of any two double lines are on a line through the meet of the other two double lines.</u> Having proved that the roots of the involution $I_{(0,1)}$ are the points of contact of the other two tangents from the meet of 0 and 1, we have only to show that the points of contact of tangents from the meet of the double lines 2 and 3 are on

the involution $\mathcal{I}_{(0,1)}$, that is to say, that the roots of $\mathcal{I}_{(2,3)}$ are in $\mathcal{I}_{(0,1)}$.

Not knowing the equation of the double line 3, in order to find the roots of $\mathcal{I}_{(2,3)}$ we make use of the well known fact, that the three catalectic sets of the fundamental involution give the three sets of two ~~pairs of~~ tangents from the meets of double lines, such as from 0 and 1, and 2 and 3. The fundamental involution of the quartic given by (1) takes the form

$$(2)\quad |a_1 b_1 b_2 c_2|\, t^4 + |a_1 b_1 c_2^2|\, t^3 + |b_2 c_2 c_1^2|\, t + \lambda\left(|a_1 b_1 a_2^2|\, t^3 + |b_2 c_2 a_1^2|\, t - |a_1 b_1 b_2 c_2|\right) = 0,$$

where $|a_1 b_1 b_2 c_2|$ denotes the determinant

$$\begin{vmatrix} a_1 b_1, & a_2 b_2 \\ b_1 c_1, & b_2 c_2 \end{vmatrix} \quad \text{and} \quad |a_1 b_1 c_2^2| = \begin{vmatrix} a_1 b_1, & a_2 b_2 \\ c_1^2, & c_2^2 \end{vmatrix},$$

such as pre

Writing down the J_8 of (1) we have

$$(8)\quad \begin{vmatrix} |a_1 b_1 b_2 c_2| & , |a_1 b_1 c_2^2| + \lambda |a_1 b_1 a_2^2| & , \quad 0 \\ |a_1 b_1 c_2^2| - \lambda |a_1 b_1 a_2^2| & , \quad 0 & , |b_1 c_1 c_2^2| + \lambda |b_1 c_1 a_2^2| \\ 0 & , |b_2 c_2 c_1^2| + \lambda |b_2 c_2 a_1^2| & , \lambda |a_1 b_1 b_2 c_2| \end{vmatrix} = 0$$

This is a cubic in λ whose roots are at once found to be $-\frac{c_1^2}{a_1^2}$, $-\frac{c_2^2}{a_2^2}$, and $-\frac{(b_1 c_2 - b_2 c_1)^2}{(a_1 b_2 - a_2 b_1)^2}$.

If we put $\lambda = -\frac{c_1^2}{a_1^2}$ in (8) we get a quartic which gives the points of contact of the pair of tangents from (0,1), and of the pair from (2,3). We mean by (01) the **** of the double lines 0 and 1. Putting $\lambda = -\frac{c_2^2}{a_2^2}$ in (8) we get the two pairs of tangents from (0,2) and (1,3), while $\lambda = -\frac{(b_1 c_2 - b_2 c_1)^2}{(a_1 b_2 - a_2 b_1)^2}$ we get the two pairs from (1,2) and (0,3).

Substituting $-\frac{c_1^2}{a_1^2}$ for λ in (8) we get, after removing the factor $b_1(a_1c_2 - a_2c_1)$,

(11) $a_1^2 b_2 t^4 + a_1(a_1c_2 + a_2c_1)t^3 - c_1(a_1c_2 + a_2c_1)t - b_2c_1^2 = 0.$

This factors into

(12) $(a_1t^2 - c_1)(a_1b_2t^2 + (a_1c_2 + a_2c_1)t + b_2c_1) = 0,$

the first factor giving the points of contact of tangents from (0,1) and the second factor giving the points of contact of the two tangents from (2,3) that is the double points of $\delta_{(2,3)}$.

We now want to show that the roots of

(13) $a_1b_2t^2 + (a_1c_2 + a_2c_1)t + b_2c_1 = 0$

are in $I_{(0,1)}$. The only condition necessary is that the product of the roots shall be $\frac{c_1}{a_1}$, and this is obviously so in the case of (13).

We get $\delta_{(2,3)}$ by polarizing (13).

(14) $\delta_{(2,3)} \equiv 2a_1b_2 t t_2 + (a_1c_2 + a_2c_1)(t + t_2) + 2b_2c_1 = 0.$

It is then also readily seen that

the roots of $I_{(0,1)}$ are a set of $I_{(2,3)}$. The fact that the double points of $I_{(0,1)}$ are on $I_{(2,3)}$ and also the double points of $I_{(2,3)}$ are a set of $I_{(0,1)}$ says the two involutions are commutative that is $I_{(0,1)}I_{(2,3)} = I_{(2,3)}I_{(0,1)}$. Thus there are a simple infinity of four-points on the curve for which $(0,1)$ and $(2,3)$ are diagonal points. That is to say choose any point t_1 of the curve, and thus determine t_2 as a set of $I_{(0,1)}$; ~~choose~~ ~~any~~ determine the other point t_3 as a set of $I_{2,3}$ and then ~~then~~ determine~~ing~~ its partner t_4. Then we say these four points will pair off, t_1 with t_3 say, and t_2 with t_4 to form sets of $I_{(2,3)}$. (See fig. 1)

If we allow this ~~four~~ point to run around the curve, $(0,1)$ and $(2,3)$ will be two fixed diagonal

(The third double line is at infinity

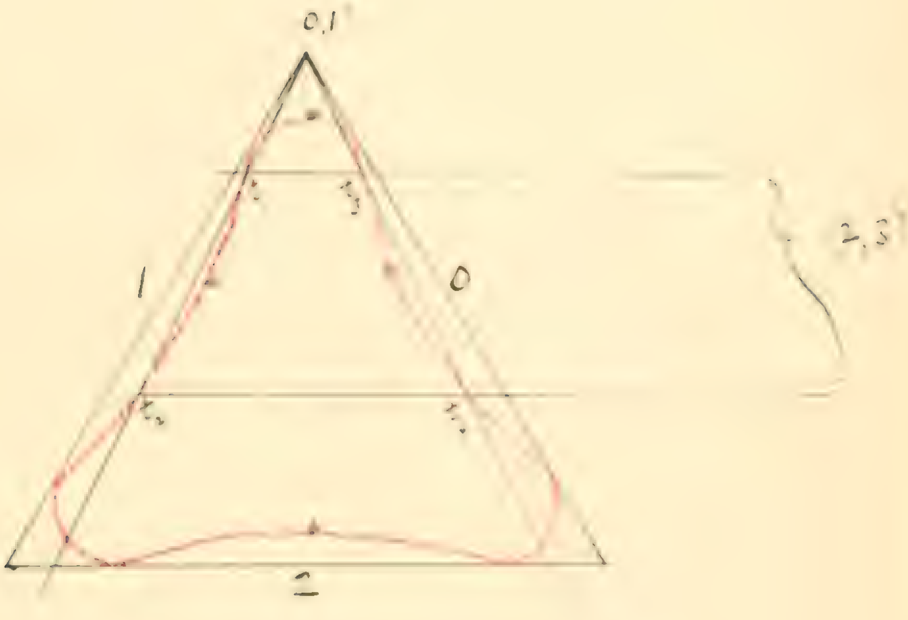


Fig. 1

points and the third diagonal point will have a locus. There will be three such loci corresponding to the three ways in which we may pair off the double lines. This question will be taken up in a subsequent paragraph of this paper.

We have proved that the points of contact of the two remaining tangents from the meet of any two double lines lie on a line through the meet of the other two. There are then six such lines, and we shall prove that they are on four points. We shall denote by $L_{(0,1)}$ the line on the points of contact of tangents from $(0,1)$, that is from the meet of the double lines 0 and 1. The other six lines are similarly named.

It is obvious by looking at a symmetrical figure with one double line at infinity that the three lines, on a point of movement, must be such as $I_{(0,1)}$, $I_{(0,2)}$ and $I_{(0,3)}$. To get these three lines we need the double points of $J_{(0,1)}$, $J_{(0,2)}$ and $J_{(0,3)}$. We have found the double points of $J_{(0,1)}$ to be

(4) $a_1 t^2 - c_1 = 0$

From symmetry the double points of $J_{(0,2)}$ are

(5) $a_2 t^2 - c_2 = 0$.

In order to get the double points of $J_{(0,3)}$, we first find $J_{(1,2)}$ and then again make use of the established rules of the fundamental involution.

Any line on the set of the double lines 1 and 2 is from (1) of the form

(7) $(a_1 t^2 + 2b_1 t + c_1)^2 - \lambda^2 (a_2 t^2 + 2b_2 t + c_2)^2 = 0$,

one factor of which is

(18) $(a_1t^2+2b_1t+c_1)-\lambda(a_2t^2+2b_2t+c_2)=0$

If t_1 be a root of (18) we have

$$\lambda = \frac{a_1t_1^2+2b_1t_1+c_1}{a_2t_1^2+2b_2t_1+c_2}.$$

Putting this value of λ in (18) and removing the factor $t-t_1$ we get

(19) $\phi_{(1,2)} \equiv (a_1b_2-a_2b_1)tt_1+(a_1c_2-a_2c_1)(t+t_1)+2(b_1c_2-b_2c_1)=0.$

Hence the double point of $\phi_{(1,2)}$ are given by

(20) $(a_1b_2-a_2b_1)t^2+(a_1c_2-a_2c_1)t+(b_1c_2-b_2c_1)=0.$

If in the fundamental involution we substitute $-\frac{(b_1c_2-b_2c_1)^2}{(a_1b_2-a_2b_1)^2}$ for λ, we get,

after removing the factor $b_1b_2(a_1c_2-a_2c_1)$,

(21) $(a_1b_2-a_2b_1)^2t^4+(a_1^2b_2c_2+a_2^2b_1c_1-a_1a_2b_2c_1-a_1a_2b_1c_2)t^3$
$+(a_1b_2c_1c_2-a_2b_1c_1c_2-a_1b_1c_2^2-a_2b_2c_1^2)t-(b_1c_2-b_2c_1)^2=0,$

which we have seen gives the double-points of $\phi_{(0,3)}$ and $\phi_{(1,2)}$.

Hence (20) is a factor of (21), and the remaining factor is

(22) $(a_1 b_2 - a_2 b_1)t^2 - (b_1 c_2 - b_2 c_1) = 0$.

and this must be the double points of $S_{(0,3)}$.

Now the three lines wanted are in each case on parameters t and $-t$; that is the double points of the three involutions under consideration are given by quadratics with no middle term. We shall then write down the line joining the parameters t and $-t$, and then substitute the value of t in each of the three cases.

From equation (1) we get the line joining t and $-t$ as the determinant

$$(23)\quad \begin{vmatrix} x_0 & x_1 & x_2 \\ a t^2 & (a_1 t^2 + 2b_1 t + c_1)^2 & (a_2 t^2 + 2b_2 t + c_2)^2 \\ a t^2 & (a_1 t^2 - 2b_1 t + c_1)^2 & (a_2 t^2 - 2b_2 t + c_2)^2 \end{vmatrix} = 0.$$

Developing and removing ~~the~~ factors we get the lines

(24) $\gamma_0\big[a_1 a_2 (a_2 b_1 - a_1 b_2) t^6 + \{a_2^2 b_1 c_1 - a_1^2 b_2 c_2 - 2a_1 a_2 b (c_1 - b_2 c)$

$+ 2 b b_2 (a_1 b_2 - a_2 b_1)\} t^4 - \{a_1 b_1 c_2^2 - a_2 b_2 c^2 + 2 c c_2 (a_2 b_1 + a_1 b_2)$

$+ 2 b b_2 (b_2 c - b_1 c_2)\} t^2 + c_1 c_2 (b_1 c_2 - b_2 c_1)\big]$

$+ \gamma_1 [a_2 a_2 t^4 - b_2 c_2 t^2] - \gamma_2 [a_1 b_1 t^4 + b_1 c_1 t^2] = 0$

If $t^2 = \frac{c}{a}$ (24) becomes

(25) $L_{(0,1)} \equiv \gamma_0 [a_1^2 b_1 c_2^2 - a_2^2 b_2 c_1^2 - 2 a^2 b_2 c_1 c_2$ [illegible]

$+ 2 a_1 a_2 b c_1 c_2 - 2 a b b_2^2$ [illegible]$]$

$+ 2\gamma_1 [b_2 (a_1 c_2 - a_2 c_1)] - \gamma_2 [a_2 b c] = 0$

If $t^2 = \frac{c_2}{a_2}$ (24) becomes

(26) $L_{(0,2)} \equiv \gamma_0 [-a_1^2 b_2 c_2^2 - a_2^2 b_2 c_1^2 + 2 a_2^2 b_1 c_1 c_2 + 2 a_1 a_2 b_1 c_2^2$

$- 2 a_1 a_2 b_2 c_1 c_2 - 4 a_2 b b_2 c_1 + 2 a b b_2 c_2 + 2 a_2 b b_2 c]$

$+ 4\gamma_1 [a_2 b_2 c_2] - 2\gamma_2 [b_1 (a_1 c_2 + a_2 c_1)] = 0$

If $t^2 = \dfrac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}$ (24) becomes

(27) $L_{(0,3)} \equiv \gamma_1 b_2^2 - \gamma_2 b_1^2 = 0$

If $L_{(0,1)}, L_{(0,2)}, L_{(0,3)}$ are on a point

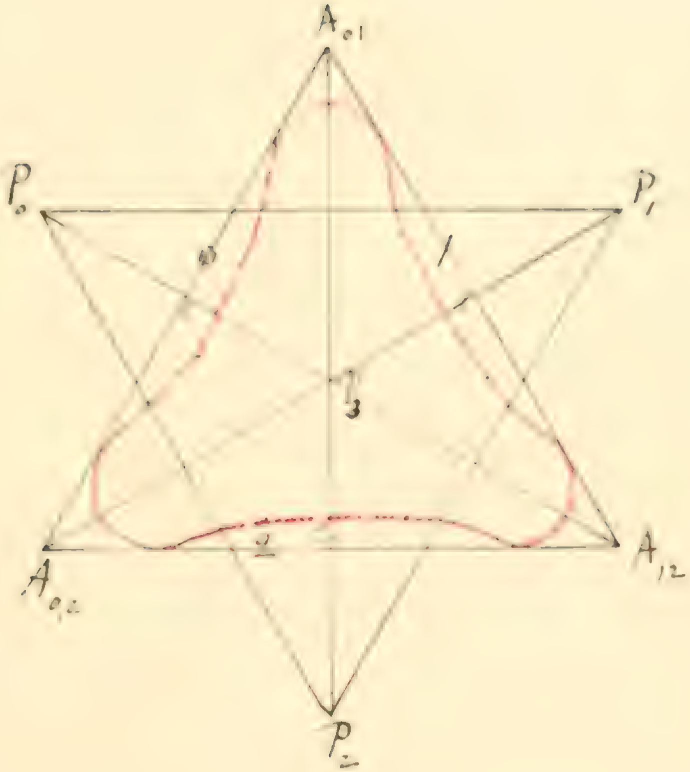
the determinant of their coefficients must vanish. The determinant may be written

$$\begin{vmatrix} \text{[illegible]} & , & \text{[illegible]} & , & \text{[illegible]} \\ \text{[illegible]} & , & \text{[illegible]} & , & \text{[illegible]} \\ 0 & , & b_2 & , & b_1 \end{vmatrix}$$

This reduced is readily seen to vanish which proves the theorem that the six lines on the points of contact of tangents from the meets of any two double lines form a complete four point.

These six lines together with the four double lines form a Desargues' Configuration 10_3. That is we have two triangles ~~two triangles~~ perspective from a point and having homologous sides meeting in three collinear points.

If we consider the symmetrical figure of the rational quartic in which the six flexes are real and one double line is at infinity we can see the configuration B. Suppose the double lines are 0, 1, 2, with 3 at infinity. Let the points (0,1) and so on be denoted by A_{01}, and so on. Let the six lines meet in the points P_0, P_1, P_2, P_3. Then we have the triangles $P_0 P_1 P_2$ and $A_{01} A_{02} A_{12}$ perspective from the point P_3, and having their homologous sides meeting in the points A_{03}, A_{13}, A_{23}; these points on the fourth double tangent 3.



A_{01}
P_0
P_1
0
1
P_3
2
A_{02}
A_{12}
P_2

We shall now study the four points more in detail. The one point obtained was that determined by the lines $l_{(01)}$, $l_{(02)}$ and $l_{(03)}$. Thus this point is paired off with the double line 0. In the same way each of the four points is paired with a double line. Now there is reason to believe that these points are in some way related to the stable conic N, which is the locus of the flex lines of cubic osculants of the rational quartic. We shall show that the four points are the poles each of the four double lines as to the conic N.

If the quartic is written

(28) $$x_i = a_i t^4 + 4 b_i t^3 + 6 c_i t^2 + 4 d_i t + e_i \,,$$

it is known that N takes the form

(29) $-3(bcx)(cdx) + 2(adx)(cdx) - 2(bcx)(bcx)$

$+ 4(bdx)^2 + (acx)^2 + 2(abx)(dcx) - 3(adx)(bcx) = 0$

where $(bcx) = \begin{vmatrix} b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \\ x_0 & x_1 & x_2 \end{vmatrix}$ etc.

Taking the quartic as given by (1) N is

(30) $x_0^2\Big[-4|a_1b_1(a_2c_2+2b_2^2)||b_2c_2(a_1c_1+2b_1^2)|$

$-4|b_2c_2(a_1c_1+2b_1^2)||a_1^2b_2c_2| - 4|a_1b_1(a_2c_2+2b_2^2)||a_1b_1c_2^2|$

$+4|a_1b_1b_2c_2|^2 + |a_1^2c_2^2|^2 + 4|a_1^2a_2b_2||b_1c_1c_2^2|$

$-8|a_1^2b_2c_2||a_1b_1c_2^2|\Big]$

$+x_1^2[10a_2b_2^2c_2] + x_2^2[10a_1b_1^2c_1]$

$+x_1x_2[-8b_2(a_1c_2+a_2c_1)] + x_0x_2\Big[-8b_1|a_1b_1(a_2c_2+2b_2^2)|$

$+8a_1b_1|b_2c_2(a_1c_1+2b_1^2)| + 8b_1c_1|a_1^2b_2c_2| - 8a_1b_1|a_1b_1c_2^2|\Big]$

$+x_0x_1\Big[8b_2c_2|a_1b_1(a_2c_2+2b_2^2)| - 8a_2b_2|b_2c_2(a_1c_1+2b_1^2)|$

$-8b_2c_2|a_1^2b_2c_2| + 8a_2b_2|a_1b_1c_2^2|\Big] = 0,$

where $|a_1b_1(a_2c_2+2b_2^2)| = a_1b_1(a_2c_2+2b_2^2) - a_2b_2(a_1c_1+2b_1^2)$,

and so on.

The coordinates of the point in question are found by getting the intersection of any two of the three lines, say

$L_{(01)}$ and $L_{(03)}$. We have at once

$$(31)\begin{cases} x_0 = 2 b_1 b_2 (c_1 | a b_2 | - c_2 | b c_1 |) \\ x_1 = b_2^{2} [2 c_1 | a_1 b_1 a_2 c_2 + 2 b_2^{2} |] + 2 b_1 b_2 | a_2 c_1 | + a_1^2 c_2 | b_1 c_2 | - c_1 a_1^2 b_2 c_1 | \\ x_2 = b_2^{2} [2 c_1 | a_1 b_1 a_2 c_2 + 2 b_2^{2} |] + 2 b_1 b_2 | a_2 c_1 | + a_1^2 c_2 | b_1 c_2 | + c_1 | a_1^2 b_1 c_1 | \end{cases},$$

where the expressions within the bars have the same meaning as in (30).

The question now is whether this point and the double line 0 are pole and polar with regard to N. To prove this we only have to find the derivative of N as to x_1 and then as to x_2, and see if the two resulting lines pass through the point represented by (31).

$$(32)\ D_{x_1} N \equiv x_0 [b_2 c_2 | a_1 b_1 a_2 c_2 + 2 b_2^{2} | - a_2 b_2 | b_2 c_2 a_1 c_1 + 2 b_1^{2} | - b_2 c_2 | a_1^2 b_2 c_2 | - a_2 b_2 | a_2 c_1 |] + x_1 [+ a_2 b_2^{2} c_2] + x_2 [- 2 a_1 b_1 b_2 c_2 - 2 a_2 b_1 b_2 c_1] = 0.$$

Taking the derivative of N as to x_2 we have

$$(33)\quad \Delta_2 P \equiv b_2\left[-b_1^2 e_1 \text{[illegible]} \right] + b_1 e_1\left(a_1^2 b_2 e_2\right) - a_1 b_1\left(a_1 b_1 e_2^2\right)\Big]$$
$$- c_1\left[-2a_1 b_1 b_2 e_2 - 2a_2 b_1 b_2 e_1\right] + c_2\left[-a_1 b_1^2 e_1\right] = 0$$

Substituting the coordinates of the point given by (30) in (32) and (33) we find that each of them is satisfied, and the theorem is proved.

We have seen that there is a single infinity of four points on the rational plane quartic for which (0,1) and (2,3) are first diagonal points and we now want to find the locus of the third diagonal point.

Dr. F. R. Conner has suggested and kindly given me the proof, that the projection of the intersection of two circular cylinders, touching and intersecting at right angles, is a general rational quartic. The general rational plane quartic may be considered as a projection of a space quartic with a node. A quartic in space is the intersection of two quadric cones. Let A and B be the vertices of two quadric cones on the curve. Choose the plane at

infinity as a plane on A and B. This plane meets each one in a pair of lines. Take the absolute as a cone touching the four lines and apolar to the pair of points A and B. Then the curve is the intersection of two circular cylinders touching and intersecting at right angles.*

We shall denote this space quartic with one node by the symbol S^4, and the plane rational quartic into which S^4 projects by R^4.

Now consider S^4. Take the line normal to the two cylinders at the node. The osculating planes of the two branches through the node contain this line. There is a single infinity of planes on this line each of which meets S^4

* Cf Marletta: Studio geometrico della quartica gobba razionale. Annali di Mat. Serie 3 Vol. 8 pp. 44.

in two points other than the node. We have thus a quadratic involution. Its double points will be given when the plane osculates one of the branches at the node; that is to say the nodal branches give the double points of the involution. That tells us that all these planes cut out from S^4 quadratics similar to the quadratic giving the node.

Projecting from a point M of space, the two tangent planes to each cylinder from M go into the four double lines of R, thus giving two pairs of double lines. The involutions, which we have previously discussed, on the units of pairs of double lines of R^4 are cut out of

by the generators of the cylinder. Hence the double points of such an involution are given by the generators that touch S^4. It is then easy to see that the parameters of the isolated node are apolar to the double points of each involution, that is the Jacobian of these pairs of points gives the nodal parameters. Also it is obvious that the double points of one involution are a set of the other, and that the node is in both involutions. This shows again the pairing off of the double lines by choosing a node.

The involutions on the meets of double lines of R^4 are the projections of points cut out on S^4 by the generators of the cylinders.

We have on S^2 a system of corners of rectangles cut out by planes on T_1 and T_2; these groups of four points through project into the groups of four points on S^2 obtained from these pairs of involutions such as $i_{1,2}$ and $i_{2,3}$. We may remark in passing that it is obvious that the parameters of the groups of four points of which we are speaking are a double of quartics —precisely a syzygetic double, being built on a quartic and its Hessian as it is seen from the space figure that the double contains three perfect squares.

The diagonals of the rectangles intersect in a point whose locus is just that of the third figure

point which we started out to find. It is seen that the locus is a line, and is the normal to the cylinders at the node. This line bisects [illegible] a line in the plane which passes through a node and cuts out a pair of parameters from R^2 harmonic to the nodal parameters. It is evident that there is only one such line for each node. We thus get three such lines and Professor Morley has shown in his lectures that these three lines meet in a point. We have the theorem: If the real section of the double locus of R^2 [illegible] three pairs of real points on R^2 with the [illegible] of [illegible] of double lines as [illegible] nodal points and [illegible] that [illegible] point [illegible] a line on the [illegible] node and [illegible] the harmonic [illegible].

Section III

The Case with three Flex Tangent to a Point.

It is well known that the locus of lines which cut a rational plane quartic in sets of four self-polar points is a conic f_2. In particular the flex tangents are such lines, hence the six flex tangents touch the f_2 conic. Now if f_2 breaks up, it breaks up into two points; then three flex tangents are on one of these points and three are on the other. It is necessarily three and three because each flex tangent counts for two tangents, and since the quartic is of class six not more than three flex tangents can be on a point.

First, let a general quartic be referred to two of its flex tangents and the line joining the two flexes

which have the parameters 0 and ∞.

$$(1)\quad \begin{cases} \psi_0 = at^4 + 4bt^3 \\ \psi_1 = 4bt^3 + 6ct^2 + 4dt \\ \psi_2 = 4dt + e \end{cases}$$

We want another flex tangent to be on the meet of ψ_0 and ψ_2. It must be of the form $\psi_0 + \lambda\psi_2$. We have then

$$(2)\quad at^4 + 4bt^3 + 4d\lambda t + e\lambda \equiv (at-\alpha)(t-\beta)^3.$$

We find upon equating coefficients that $\alpha = 2b$, $\beta = -\frac{2b}{a}$ and

$$\lambda = -\frac{4b^3}{a^2 d} = -\frac{4b^4}{a^3 e}.$$

Hence $ae = bd$.

The third flex tangent is then

$$(3)\quad (at-2b)\left(t+\frac{2b}{a}\right)^3 = 0$$

Now if we let $a = 2b$ we have only chosen the unit point, i.e. $t=-1$ is a flex point of the curve. If $a = 2b$ then $e = 2d$. The flex tangent is now

$$(4)\quad (t-1)(t+1)^3 = 0$$

This shows at once that the three points in which the three flex tangents meet the curve again are the cubic covariant of the three flexes, for which the parameters are $t = 0, 1$ and ∞.

The present curve has the equations

$$(5)\quad \begin{cases} z_0 = 2bt^4 + 4bt^3 \\ z_1 = 4bt^3 + 6ct^2 + 4dt \\ z_2 = 4dt + 2d \end{cases}$$

What has just been stated is this: the flex tangent z_0 meets the curve again at $t = -2$, the flex tangent z_2 meets the curve again at $t = -\frac{1}{2}$; the other flex tangent on the meet of z_0 and z_2 meets the curve again at $t = 1$. The cubic giving these three parameters is

$$(6)\quad 2t^3 - 3t^2 - 3t - 2 = 0.$$

And the cubic giving the flexes $0, \infty, -1$ is

$$(7)\quad t^2 + t =$$

Then we say (6) is the Cubicovariant of (7), that is (6) is the Jacobian of (7) and its Hessian. The Hessian of (7) is

$$\begin{vmatrix} t, & t+1 \\ t+1 & 1 \end{vmatrix} = 0, \text{ or}$$

(8) $t^2+t+1 = 0$

The Jacobian of (7) and (8) is

$$\begin{vmatrix} 2t+1, & t^2+2t \\ 2t+1, & t+2 \end{vmatrix} = 0, \text{ or}$$

(6) $2t^3+3t^2-3t-2 = 0$,

and this is precisely (6) which was to be shown.

We shall now find the cubic giving the parameters of the other three flexes. We know the flex cubic is given by the Jacobian of the two members of the fundamental involution. Calculating the fundamental involution of our quartic given by (5)

we have the two members

(10) $ct^4 + 2et^3 + 2dt = 0$.

(11) $2bt^2 + 2ct + e = 0$.

The Jacobian of (10) and (11) is

$$\begin{vmatrix} 2ct^3 + 3et^2 + 2d & ct^4 + 2et^2 \\ 2bt + e & bt^2 + 3et + 2d \end{vmatrix} = 0$$

or with e coming out as a factor

(12) $2bt^5 + (3c + 2b)t^4 + 2et^3 + (3c + 2d)t^2 + 2dt = 0$

This factors into (7) and

(13) $2bt^3 + 3ct^2 + 2et + 2d = 0$.

the latter giving the second set of flex parameters.

Suppose the g_2 conic breaks up into the points p and p'; that is the flex tangents represented by (7) meet at p and those by (13) at p'. Then any point on the line pp', which we shall call the g_2 line, will be of the form $p - \lambda^2 p'$. Now the

six tangents from the point p are given by the square of a cubic say $[(\alpha t)^3]^2$ and likewise the six tangents from q are given by $[(\beta t)^3]^2$. Then the six tangents from any point of the pq line will be

$$[(\alpha t)^3]^2 - \lambda^2 [(\beta t)^3]^2 = 0,$$

which factors into

$$[(\alpha t)^3 + \lambda(\beta t)^3][(\alpha t)^3 - \lambda(\beta t)^3] = 0$$

showing that the tangents all along that line pair off into two sets of three. We have then a pencil of cubics which is

$$(2) \quad 2bt^3 + (3c+\lambda)t^2 + (3c+\lambda)t + 2d = 0.$$

The pencil is a set of apolar cubics as may be seen by writing the apolarity condition of (3) and (2). We have

$$2bd - c(3c+\lambda) + c(3c+\lambda) - 2bd \equiv 0$$

We shall now find the g_2 conic in order to find the four points in which the g_2 line meets the curve. Cut the curve (5) by any line (ξx). We have then

(5) $2b\xi_0 t^4 + 4b(\xi_0+\xi_1)t^3 + 6c\xi_1 t^2 + 4d(\xi_1+\xi_2)t + 2d\xi_2 = 0$

If this is a self-apolar section we have

(6) $4bd\xi_0\xi_1 - (4bd-3c^2)\xi_1^2 - 4bd\xi_1\xi_2 = 0$

This is the g_2 conic which factors into

(7) $\xi_1 = 0$ and

(8) $4bd\xi_0 + (4bd-3c^2)\xi_1 + 4bd\xi_2 = 0$

That is the coordinates of the point p are $(0, 1, 0)$, and p', $(4bd, 4bd-3c^2, 4bd)$. Therefore the g_2 line has the equation

$x_0 - x_2 = 0$,

or it meets the curve in the quartic*

(9) $bt^4 + 2bt^3 - 2dt - d = 0$

We shall now prove that if ...

* When f_2 breaks up there is then one line section in the fundamental involution and it is here seen that it is the g_2 line.

The Jacobian of the system of involutions.

Taking the Jacobian of (17) and (18) we get

$$\begin{vmatrix} 2t+1 & , \quad t^2+2t \\ 2bt^2+2ct+c & , \quad ct^2+2ct+2d \end{vmatrix} = 0, \text{ or}$$

(20) $t^4+2bt^3-2ct-d=0$,

which is identical with (19).

This fact enables us to draw a conclusion about the pairing of the tangents from the points where the J_2 line meets the curve. At these points the tangent counts for two, and there are four others. Hence the pairing must be one of two ways. Either the tangent at the point is paired with two of the four, and this same tangent with the other two of the four; or the tangent at the point taken twice is paired with one of the four, leaving the other

three for a set. That the latter is the case is seen from the fact that these four points in which the j line meets the curve are the Jacobian of the system of cubics.

The system of cubics has a unique apolar quartic and its Hessian is the Jacobian of the cubics.

Write any quartic

(21) $a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4 = 0$.

If the cubic given by (7) is apolar to (21) the result of operating with (7) on (21) must be identically zero; the same is true of the cubic given by (13).

Writing (7) and (13) homogeneously and putting $\frac{\partial}{\partial t_2}$ for t_1 and $-\frac{\partial}{\partial t_1}$ for t_2, we have

(22) $-\frac{\partial}{\partial t_1}\frac{\partial^2}{\partial t_2^2} + \frac{\partial^2}{\partial t_1^2}\frac{\partial}{\partial t_2}$, and

(23) $2k\frac{\partial^3}{\partial t_2^3} - 3c\frac{\partial^2}{\partial t_2^2}\frac{\partial}{\partial t_1} + 3c\frac{\partial^2}{\partial t_1^2}\frac{\partial}{\partial t_2} - 2k\frac{\partial^3}{\partial t_1^3}$

Operating, as indicated by (22) and (23), on (21) and making the two results identically zero we get four equations from which we can determine the a's in terms of the coefficients of the cubics. The quartic (21) turns out to be

(24) $f^2t^4 + 4fdt^3 + 6bdt^2 + 4bdt + d^2 = 0$

It is readily seen that the Hessian of (24) is the Jacobian of the system of cubics given by [illegible] $= 0$.

It is known that, having a system of apolar cubics and the apolar quartic, if a root of the Hessian of the quartic is a double root of one of the cubics of the system then the other root of the cubic is a root of the Steinerian

Now at the points where the g_2 line meets the curve we have seen that two of the roots of the cubic become equal, that is a tangent counted twice is paired with a single tangent. Since this double root is a root of the Hessian of (24), the parameter of the point of contact of the single tangent must be a root of the Steinerian of (24). The Steinerian of a quartic f is known to be of the form

$$g_3 f + 1\,g_2 H = 0,$$

where g_2 and g_3 are the invariants of f, and H its Hessian. But g_2 of the quartic (24) is zero. Hence the Steinerian is the quartic itself [illegible]

We assert further that the quartic (24)

is the quartic of which the system of cubics are first polars.

Salmon tells us how to find the quartic when the cubics are given.*

It is $12H(J) + j_2 J$, where J is the Jacobian of the cubics, $H(J)$ the Hessian of the Jacobian, and j_2 the apolarity condition of J. But in our case

$$J \equiv bt^4 + 2bt^3 - 2dt - d = 0,$$

and $j_2 = 0$. We have left only the Hessian of J which is

$$b^2t^4 - 4bdt^3 + 6bdt^2 - 4bdt + d^2 = 0,$$

and this is again just (24).

To sum up, we have then that the quartic (24) is the quartic to which the system of cubics is apolar, the quartic of which the system are first polars, and besides it is its own

* Salmon, Lessons on Higher Algebra, § 217

Steinerian and gives the points of contact of the four tangents which are such with a tangent conic line that is with the tangents at the points where the g line meets the curve.

We have now a pencil of cubics which are the first polars of a infinite quartic, and thus we get a sort of mapping of the cubics on to the plane. To any cubic we have a definite corresponding point P of the curve that is the point with regard to which the cubic is the polar of (24). But paired with that cubic is a second one, and we get a point P_2 on the curve from it in a

similar way. This is then a quadratic involution set up on the curve. Since the two cubics come together when we reach the points p and p', then $t_1 = t_2$, that is the two points of I_2 correspond to the double points of the involution.

We shall now find the quadratic giving the double points of the involution. The coefficients of the polarized quartic must be proportional to the coefficients of the pencil of cubics. We can then determine λ in terms of t_1, and since we have chosen for our base cubics the two corresponding to the I_2 points, the double points of the involution will be given by $\lambda = 0$ and $\lambda = \infty$

Polarizing (24) with regard to t, we get

25) $(b^2 t_1 - bd)t^3 + 3(bdt_1 + bd)t^2 + 3(bdt_1 - bd)t + bdt_1 + d^2 = 0$

whose coefficients are to be propor-
tional to those of

$2bt^3 + 3(c+\lambda)t^2 + (3c+\lambda)t + 2d = 0.$

Hence we have

26) $$\lambda = \frac{(cbd - 3bc)t_1 + cbd - 3cd}{bt_1 + d}.$$

Since the double points of the invo-
lution are given by $\lambda = 0$ and $\lambda = \infty$
the product of the numerator and
the denominator of the expression
for λ gives the double points, or
we may say the numerator gives
the double point corresponding to $\lambda = 0$
and the denominator gives the
double point corresponding to
$\lambda = \infty$

We now look for the pair of double points on the curve. There is a pair of points q on the curve which seem to have some relation to the double points of the involution. If the fundamental involution of the curve is $(\alpha t)^4 + \lambda(\beta t)^4$ then q is

$$(\alpha\beta)^3(\alpha t)(\beta t) = 0$$

or in terms of the determinants, writing p_{ij} for $\begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix}$, we have

$$(27)\quad q \equiv (p_{03} - 3p_{12})t^2 + (p_{02} - 2p_{13})t + p_{12} - 3p_{23} = 0$$

The fundamental involution in our case is given by (10) and (11) and the points q take the form

$$(28)\quad (f-e)t^2 - 2et + (d-e) = 0$$

~~Operating~~ with (28) on the pencil of cubics given by (14), we have the following relation between λ and t.

(29) $\lambda = \frac{(3bc - ad)t_1 + 3cd - [illegible]}{[illegible]}$.

It is to be noticed that λ in (29) is just the negative of λ in (20). This shows a sort of cross working of the quartic of which the cubics are first polars and the quadratic q. That is, if C_1 is the polar cubic of the quartic as to t_1 and C_2 is the polar cubic as to t_2, then q operating on C_1 gives the point t_2 and q operating on C_2 gives the point t_1.

The double points of the involution are the points with regard to which the polars of the quartic 2-1 are taken in order to get the two base cubics of the pencil, that is the two flex cubics. Now at these two points we have

a tangent to the curve; that is to say each of these points belong to some cubic of the pencil, and we wish to know the relation of these cubics to the flex cubics.

Consider the double point $bt+d$ which gives the flex cubic

$$t^3 - t = 0 .$$

We proceed to find the cubic to which $bt+d$ belongs. We must have

$$2bt^3 + (3c+1)t^2 + (3c+1)t + 2d \equiv (bt+d)(a_1t^2+a_2t+a_3)$$

Equating coefficients we find a_1, a_2, a_3 are all equal and may be unity.

The cubic then to which $bt+d$ belongs is

(30) $$bt^3 + (b+d)t^2 + (b+d)t + d = 0$$

The quadratic giving the two points of contact other than $bt+d$ is

(31) $$t^2 + t + 1 = 0 ,$$

and this is at once seen to be

the Hessian of the flex cubic (31) is.
It will be remembered that the cubic covariant of the flex cubic was the extra points of intersection of the flex tangents with the curve.

Now the cubic $t^3+1=0$ which we have considered is special geometrically only; it is one of the pencil and behaves as any other member of the pencil. Any two of the three tangents given by (30) are therefore the Hessian pair of some cubic of the pencil. We look for the three cubics thus obtained. The cubic (30) written in terms of its roots is

(32) $(t+1)(t-\omega)(t-\omega^2) = 0$.

We assert that the three cubics of which the three pairs of (32) are the [illegible] the Hessian of these cubics

which are the first polars of the quartic as to the three roots of (32)

Polarizing (24) as to ω, we have

(33) $(b^2\omega + bd)t^3 + 3(b\,d\omega + bd)t^2 + 3(bd\omega + bd)t - b\,d\omega + d^2 = 0$

The Hessian of this cubic is

$$\begin{vmatrix} (b\omega + d)t + d\omega + d, & (d\omega + d)t + d\omega + d \\ (b\omega + b)t + b\omega + b, & (b\omega + b)t + b\omega + d \end{vmatrix} = 0,$$

or

(34) $bt^2 - (b\omega^2 - d)t - d\omega^2 = 0,$

which is just the two remaining roots of (32).

That is we have a system of cubics, which are the first polars of a quartic, such that if the roots of any one be t_1, t_2, t_3 then the cubic which is the polar of the quartic as to any one of the t's and is therefore one of the system, has the other two t's for its Hessian roots.

Section II

The Second Three Double
Lines on a Point

First of all we shall find the parametric equation of a quartic with a [illegible] point that is with three double lines on a point. We may choose three points of the curve; say $0, \infty, 1$. Let 0 and ∞ be the points of contact of one double tangent say

$$y_0 = t^2.$$

Let y_1 be a double tangent with points of contact 1 and α; then

$$y_1 = (t-1)^2(t-\alpha)^2.$$

If there is another double tangent on the meet of y_0 and y_1, it is of the form $y_0 + \lambda y_1 = 0$, and we must have

$$(t-1)^2(t-\alpha)^2 + \lambda t^2 \equiv (t^2 - c_1 t + c_2)^2$$

Equating coefficients we find the only possibility is $\alpha = -$ [illegible].

Hence the second double tangent is

$$y_0 = (t-1)^2(t+1)^2,$$

and the third double tangent on the meet of y_0 and y_1 is

$$y_0 + 4y_1 = 0.$$

Now we may choose the fourth double tangent generally and the equation of the curve is

$$(1)\quad \begin{cases} y_0 = (t-1)^2(t+1)^2 \\ y_1 = t^2 \\ y_2 = (t^2 - s_1 t + s_2)^2 \end{cases}$$

In order to get a better form make the following transformation:

$$y_0' = y_0 + 2y_1,$$

$$y_1' = 6y_1,$$

$$y_2' = \frac{2}{s_1}\left[y_0 - y_2 + (2 + s_1^2 + 2s_2)y_1\right].$$

Then dropping the primes the curve takes the form

$$(2)\quad \begin{cases} x_0 = t^4 \\ x_1 = [illegible] \\ x_2 = 4t^3 - [illegible] \end{cases} \qquad \left([illegible] = -\frac{[illegible]}{3} \right)$$

The three double tangents on a point are now seen to be

$x_1 = 0$,

$3x_0 + x_1 = 0$

$3x_0 - x_1 = 0$.

If we cut (2) by any line

$$(3)\quad (\xi x) \equiv \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0 ,$$

we have a quartic in t, which is

$$(4)\quad \xi_1 t^4 - 4\xi_2 t^3 - [illegible] - [illegible] = 0$$

Since we can choose the two invariants i_2 and i_3 of (4) we get a linear [illegible] of a [illegible] curve.

[illegible]

point are the Hessian lines of the other double lines on the [illegible] [illegible] of the [illegible] to the $\frac{2}{3}$ cubic from the [illegible] point are the [illegible] of the three double lines.

The three double lines are given by

(5) $[illegible] y_1^2 [illegible] - z^3 = 0$

The Hessian of (5) are

$$\left| \begin{array}{ll} [illegible] & [illegible] \\ [illegible] & [illegible] \end{array} \right| = 0$$

(6) $[illegible] y_1^2 - z^2 = 0$.

The [illegible] as the Jacobian of a cubic and its Hessian. The Jacobian of (5) and (6) are

$$\left| \begin{array}{ll} [illegible] & [illegible] y_1^2 - 3 z^2 \\ [illegible] & [illegible] \end{array} \right| = 0$$

(7) $y_1 z^2 - y_1^3 = 0$.

First writing the $\frac{?}{2}$ [illegible] we [illegible]

(8) $\xi_1^2 + 3\xi_1^2 - [illegible]\xi_2\xi_2^2 + [illegible]\xi_0\xi_2 = 0$

In order to get the tangents of $\frac{?}{2}$ from the [illegible] point [illegible] put $\xi_2 = 0$ [illegible]), [illegible] [illegible] have

(9) $\xi_1^2 - 3\xi^2 = 0$

[illegible] [illegible] [illegible] points we write [illegible] for ξ_1 and [illegible] for $-\xi_0$, and put

(10) $2x^2 - y^2 = 0$

[illegible] [illegible] the same as (9).

Next writing the $\frac{?}{3}$ of (4) we have

$$\begin{vmatrix} \xi_1, & \xi_2, & \xi_1 \\ \xi_2, & \xi_1, & \xi_2\xi_2 \\ \xi_1, & \xi_2\xi_2, & \xi_0 + m\xi_2 \end{vmatrix} = 0, \text{ or}$$

(11) $\xi_1^3 - [illegible]\xi_2^3 - \xi_0^2\xi_1 - (\xi_2^2 + 1)\xi_0\xi_2^2 + [illegible]\xi_2\xi\xi_2^2 + [illegible]\xi\xi\xi_2 = 0$.

To [illegible] the [illegible] of $\frac{?}{3}$ from the [illegible] point we [illegible] put $\xi_2 = 0$, [illegible] [illegible] [illegible]

$= S_1^3 - S_2 S_1 = 0$,

[illegible] hence this curve is

(2) $y_1^2 y_2 - y_3^2 =$

curve, or its inverse equations (1).

We shall now find the stem curve. It [illegible] is defined as the locus of the [illegible] of the osculants of the rational quartic. If the quartic is

$y_i = (a_i t)^5$ (without binomial coefficients)

then the cubic osculant at a point T is

$y_i = (T D_t + D_{t_1})(a_i t)^2$.

Transform the quartic given by (2) we have for the cubic osculant at T

$$(14)\quad \begin{cases} y_0 = T t^3 + \\ y_1 = 3t^2 + 3Tt \\ y_2 = t^3 + 3Tt^2 + 3S_2 t + S_2 T + [illegible] \end{cases}$$

But this takes by [illegible]

(15) $(u) \equiv u_0 z_0 + u_1 z_1 + u_2 z_2 = 0$ and

(16) $(v) \equiv v_0 z_0 + v_1 z_1 + v_2 z_2 = 0$.

We get the two following equations

(17) $(u_0 T + u_2)t^3 + 3(u_2 T + u_1)t^2 + 3(u_1 T + u_2 S_2)t$
$$+ (u_0 + u_2 S_2 T + m u_1) = 0$$

(18) $(v_0 T + v_2)t^3 + 3(v_2 T + v_1)t^2 + 3(v_1 T + v_2 S_2)t$
$$+ (v_0 + v_2 S_2 T + m v_1) = 0.$$

If these two lines are made to cut in sets of [illegible] points the [illegible] of these intersections must be a [illegible] line which [illegible] through the [illegible] of the [illegible].

Treating (17) and (18) above we have

(19) $(u_0 v_2 - u_2 v_0)(S_2 T^2 + mT - 1) - (u_1 v_2 - u_2 v_1)(3S_2 - 3T^2) = 0.$

Or since z is the intersection of u and v we get

(20) $z_1(3S_2 - 3T^2) - z_2(S_2 T^2 + mT - 1) = 0.$

For a given T this is a line.

the line of flexes of the cubic osculant at the point T of the quartic. For a varying T we get the locus of this line, which locus is the same curve. Its parametric equation is:

$$(21)\quad \begin{cases} \xi_0 = 3S_2 - 3T^2 \\ \xi_1 = S_2T^2 + 2mT - 1 \\ \xi_2 = 0 \end{cases}$$

Since $\xi_2 = 0$, all the flex lines of the cubic osculants pass through that point which is the syzygetic point. That is A' is the syzygetic point counted twice.

The cubic osculants at the points of contact of the double tangents have some interesting properties. Consider first the double tangent

$3y_1 - y = 0$ whose points of contact are 1 and -1. The cubic osculant at $t = 1$ is

$$(22) \quad \begin{cases} y_1 = t^3 - [illegible] \\ y = 3t^2 - 3t \\ y_2 = t^3 - 3t^2 - 3\delta_2 t + \delta_2 + m \end{cases}$$

We see at once that this curve passes through V with the parameter [illegible] of the [illegible] point of contact [illegible] the tangent [illegible] passes through V it must have a flex there. The flex tangent is

$$(23) \quad y_1 + y = [illegible]$$

The cubic osculant at $t = -1$ is

$$(24) \quad \begin{cases} y_1 = -t^3 + [illegible] \\ y = 3t^2 - 3t \\ y_2 = t^3 - 3t^2 + 3\delta_2 t - \delta_2 + m \end{cases}$$

This likewise passes through V with the parameter of the other point

of contact of the double tangent and has a flex there. The flex tangent is the same as the other cubic osculant. That is when three double lines are on a conic the cubic osculants at the two points of contact of any one of these three double lines pass through the opposite point and has a flex there and the two cubics have the same flex tangent.

Consider the double line $3p_0 - x$ [illegible] whose points of contact are i and $-i$. The cubic osculant at $t = i$ is:

$$(29)\quad \begin{cases} x_1 = it^3 + \text{[illegible]} \\ x = 3t^2 + 3it \\ x_2 = \text{[illegible]} \end{cases}$$

The flex tangent at [illegible] is

$$(30)\quad x_1 - x = 0.$$

The cubic osculant at $T = -i$ is

$$(26)\quad \begin{cases} z_1 = -2t^3 + 1 \\ z = 3t^2 - 3it \\ z_2 = \text{[illegible]} \end{cases},$$

and its flex tangent is (20).

Considering the double tangent $z_1 = 0$ whose points of contact are 0 and ∞, we have, for $T = 0$, the cubic osculant

$$(2\text{[illegible]})\quad \begin{cases} z_1 = 1 \\ z = 3t^2 \\ z_2 = \text{[illegible]} \end{cases},$$

which passes through N with the parameter ∞, and the flex tangent is

$$(2\text{[illegible]})\quad z_1 = 0.$$

For $T = \infty$ we have

$$(3\text{[illegible]})\quad \begin{cases} z_1 = t^3 \\ z = 3t \\ z_2 = \text{[illegible]} \end{cases},$$

and its flex tangent is (2[illegible]).

Incidentally the [illegible] [illegible] of our reference lines, i.e. that is $z = 0$ is a double tangent with points of contact 0 and ∞. Take the cubic osculants at these points and they have a common flex tangent, and it is the side t_∞ of our reference triangle.

Another point to be noticed is that the three flex tangents on the singular point of the cubic osculants coincide, which tangents are given by (23), (24), (25), are just the third tangents to f_3 from that point as given by (3).

We found in a previous section that the points of contact of the two other tangents from the point

of any two double tangents of the rational quartic lie on a line with the meet of the other two double tan[illegible]: furthermore we found the six such lines are [illegible] four points. Now, in the syzygetic case, three of these lines are three double lines, and the [illegible] six are on the syzygetic point.

We shall now find the equations of these six lines. We know the points of contact of tangents from the meets of double tangents are given by the catalectic sets of the fundamental involution. Calculating the fundamental involution of the quartic (2) we have

$$(31) \quad 4t^4 + 4\lambda t^3 + 4(m - \lambda g_2)t - 4 = 0.$$

The g_3 of (31) is

$$(32) \quad \lambda^2 [illegible] (m - \lambda g_2)^2 = [illegible]$$

Thus are 3 values for λ, as the roots are $\lambda = \infty$, $\frac{m}{S_2+1}$, $\frac{m}{S_2-1}$.

Substituting these values separately in (31) we have the three catalectic sets

(33) $t^3 - S_2 t = 0$,

(34) $(S_2+1)t^4 + mt^3 + mt - (S_2+1) = 0$,

(35) $(S_2-1)t^4 + mt^3 - mt - (S_2-1) = 0$.

Remembering that $m = -\frac{2(S_2^2-1)}{S_1}$, the three sets may be factored as follows:

(36) $t(t^2 - S_2) = 0$,

(37) $(t^2+1)(S_1 t^2 - 2(S_2-1)t - S_1) = 0$,

(38) $(t^2-1)(S_1 t^2 - 2(S_2+1)t + S_1) = 0$.

These six factors ~~give~~ will determine the six lines sought. Of course the three double lines are known and the first factor in each case gives the points of contact. We want now the lines that cut out the

parameters of the other factor in each case. Since the lines are on the syzygetic point they must be of the form $z_0 + \lambda z_1 = 0$. Now a line that cuts out the two parameters which are given, will cut out another quadratic on the curve, say

$$at^2 + bt + c = 0.$$

Then by equating coefficients we can determine a, b, c, and thereby λ which gives the equation of the line. We have

$$t^4 + 6\lambda t^2 + 1 \equiv (t^2 - s_2)(at^2 + bt + c),$$

from which we find $\lambda = -\dfrac{1+s_2^2}{6s_2}$.

Therefore $z_0 + \lambda z_1$ becomes

(39) $6s_2 z_0 - (s_2^2+1) z_1 = 0$.

Considering the last factor of (37) we have, while a, b, c, are not the same as before,

$$t^4 - 6\lambda t^2 + 1 \equiv (s_1 t^2 - 2(s_2-1)t - s_1)(at^2 + bt + c),$$

from which $\lambda = -\dfrac{s_1^2 + 2(s_2-1)^2}{3s_1^2}$. and

the line becomes

(40) $3S_1^2 y_0 - [S_1^2 + 2(S_2-1)^2] y_1 = 0$.

Then considering the factor of (38)

$t^4 + 6\lambda t^2 + 1 \equiv (S_1 t^2 - 2(S_2+1)t + S_1)(at^2 + bt + c)$,

from which we find $\lambda = \dfrac{S_1^2 - 2(S_2+1)^2}{3S_1^2}$

and the third line becomes

(41) $3S_1^2 y_0 + [S_1^2 - 2(S_2+1)^2] y_1 = 0$.

We shall now find the three lines from the syzygetic point to the nodes. We know the Jacobians of the factors of the three syzygetic sets give the nodes.

The Jacobian of the two factors of the set given by (36) is

(42) $t^2 + S_2 = 0$.

The Jacobian of the factors of (37) is

(43) $(S_2-1)t^2 + 2S_1 t - (S_2-1) = 0$.

The Jacobian of the factors of (38) is

(44) $(S_2+1)t^2 - 2S_1 t + S_2 + 1 = 0$.

Now the lines from the syzygetic point to the nodes will again be of the form $x_0 + \lambda y_1 = 0$. If such a line cuts out besides the node a pair of parameters given by the quadratic $at^2 + bt + c = 0$ we have the identity

$$t^4 - 6\lambda t^2 + 1 \equiv (t^2 + S_2)(at^2 + bt + c),$$

from which we can find a, b, c, and thus determine λ. In this case $6\lambda = \dfrac{S_2^2 + 1}{S_2}$, and the line is

(45) $6S_2 x_0 + (S_2^2 + 1) y_1 = 0$.

Now using (43) we have

$$t^4 + 6\lambda t^2 + 1 \equiv [(S_2-1)t^2 + 2S_1 t - (S_2-1)][at^2 + bt + c]$$

from which $6\lambda = \dfrac{-2(S_2-1)^2 - 4S_1^2}{(S_2-1)^2}$ and the line is

(46) $3(S_2-1)^2 x_0 - (2S_1^2 - (S_2-1)^2) y_1 = 0$.

From using the node given by [illegible] we get

$t^3 - a\lambda t^2 + 1 \equiv [(S_2+1)t^2 - 2S_1 t + (S_2+1)][\text{[illegible]}]$

and hence $a\lambda = \dfrac{2(S_2+1)^2 - 4S_1^2}{(S_2+1)^2}$

and the third line is

(47) $3(S_2+1)^2 z_0 - (2S_1^2 - (S_2+1)^2) z_1 = 0.$

We have now four sets of three lines on the syzygetic point, viz.,

the three double lines

$$\mathrm{I}\ \begin{cases} z_1 = 0 & (A) \\ 3z_0 + z_1 = 0 & (B) \\ 3z_0 - z_1 = 0 & (C) \end{cases}$$

the lines to g_3

$$\mathrm{II}\ \begin{cases} z_0 = 0 & (A) \\ z_1 - z_0 = 0 & (B) \\ z_1 + z_0 = 0 & (C) \end{cases}$$

the lines to the nodes

$$\mathrm{III}\ \begin{cases} 6S_2 z_0 + (S_2^2+1) z_1 = 0 & (A) \\ 3(S_2-1)^2 z_0 - (2S_1^2 + (S_2-1)^2) z_1 = 0 & (B) \\ 3(S_2+1)^2 z_0 - (2S_1^2 - (S_2+1)^2) z_1 = 0 & (C) \end{cases}$$

Lines on the points of contact of tangents from the vertex of the three double lines on the syzygetic point with the fourth double line

$$\text{II}\quad\begin{cases} 2 s_2 z_0 - (s_2^2+1) z_1 = 0 & (A) \\ 3 s_1^2 z_0 - \{2(s_2-1)^2 - s_1^2\} z_1 = 0 & (B) \\ 3 s_1^2 z_0 - \{2(s_2-1)^2 - s_1^2\} z_1 = 0 & (C) \end{cases}$$

We have already found that the second set is the Cubicovariant of the first set. Now we say the lines marked (A) in the four sets are harmonic and so are those marked (B) and (C).

To prove this we only have to show that the four lines are of the form

$\alpha x = 0$, $\beta y = 0$, $\alpha x + \beta y = 0$, $\alpha x - \beta y = 0$,

that is the parameters $0, \infty, 1, -1$ are harmonic. The set (A) are obviously of this type. To see that the set (B)

[illegible] of the same table [illegible] let $y_0 - y_1 = y_0$
$y_2 - y_0 = y_1$ $s_2 - [illegible] = a$ then we have

$$(B)\begin{cases} y_1 = 0 \\ y_0 = 0 \\ y_1 - \dfrac{3a+3s_1^2}{a-s_1^2}\,y_0 = 0 \\ y_1 + \dfrac{3a-3s_1^2}{a-s_1^2}\,y_0 = 0 \end{cases}$$

Hence the set (B) are harmonic.

In the set (C) let $3y_0 - y_1 = y_1$ $-y_0 = y_0$
$(s_2+1)^2 = b$, and we have

$$(C)\begin{cases} y_1 = 0 \\ y_0 = 0 \\ y_1 + \dfrac{3b-3s_1^2}{b+s_1^2}\,y_0 = 0 \\ y_1 - \dfrac{3b-3s_1^2}{b+s_1^2}\,y_0 = 0 \end{cases}$$

Hence we have three sets of harmonic lines on the syzygetic [illegible]

We may see that the study of the syzygetic case of the rational quartic is the same as that of a conic and four points such that conics on them cut out a syzygetic set from the given conic. For if we call the nodes A_0, A_1, A_2 and the syzygetic point K and then convert the quartic into a conic α by the transformation $y_i = 1/x_i$ the four points all behave alike. The double lines become conics which touch the conic α twice. Three of these meet at K. The conic on A_0, A_1, A_2 and tangent to α would correspond to the fourth double line. We have

four points d_3, d, d_2, V, such that conics on them cut out a syzygetic series from α, that is such that three conics can be drawn on them bitangent to α.

Biographical Note.

Thomas Bryce Ashcraft was born at Marshville, Union County, North Carolina, November 27, 1882. He was prepared for college at the Wingate High School. In the Fall of 1903 he entered Wake Forest College, from which he was graduated in 1906 with the degree of Bachelor of Arts. In October [illegible] he entered the Johns Hopkins University with Mathematics as his principal subject, and Physics and Astronomy as first and second subordinates respectively. He held a North Carolina Scholarship during the years [illegible].

He wishes to thank Doctors Morley, Cohen, Coble, Hulburt, Bliss, Pfund, and Anderson for valuable instruction and encouragement in his university course. He also expresses his sincere gratitude to Professor Morley for his helpful suggestions and constant inspiration in the preparation of this dissertation.